Remoteness, proximity and few other distance invariants in graphs

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Abstract

We establish maximal trees and graphs for the difference of average distance and proximity proving thus the corresponding conjecture posed in [4]. We also establish maximal trees for the difference of average eccentricity and remoteness and minimal trees for the difference of remoteness and radius proving thus that the corresponding conjectures posed in [4] hold for trees.

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1 Introduction

All graphs G in this paper are simple and connected. Vertex set of graph G will be denoted with V, edge set with E. Number of vertices in G is denoted with n, number of edges with m. A path on n vertices will be denoted with P_n, while C_n will denote a cycle on n vertices. A *tree* is the graph with no cycles, and a *leaf* in a tree is any vertex of degree 1.

The *distance* $d(u,v)$ between two vertices u and v in G is defined as the length of a shortest path connecting vertices u and v. The average distance between all pairs of vertices in G is denoted with $\bar{l}$. The *eccentricity* $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G. The *radius* r of graph G is defined as the minimum eccentricity in G, while the *diameter* D of G is defined as the maximum eccentricity in G. The average eccentricity of G is denoted with ecc. That is

$$r = \min_{v\in V} e(v),\ D = \max_{v\in V} e(v),\ ecc = \frac{1}{n}\sum_{v\in V} e(v).$$

The *center* of a graph is a vertex v of minimum eccentricity. It is well-known that every tree has either only one center or two centers which are adjacent. The *diametric path* in G is a shortest path from u to v, where $d(u,v)$ is equal to the diameter of G.

The *transmission* of a vertex v in a graph G is the sum of the distances between v and all other vertices of G. The transmission is said to be *normalized* if it is divided by $n-1$. Normalized transmission of a vertex v will be denoted with $\pi(v)$. The *remoteness* ρ is defined as the maximum normalized transmission, while *proximity* π is defined as the minimum normalized transmission. That is

$$\pi = \min_{v \in V} \pi(v), \ \ \rho = \max_{v \in V} \pi(v).$$

In other words, proximity π is the minimum average distance from a vertex of G to all others, while the remoteness ρ of a graph G is the maximum average distance from a vertex of G to all others. These two invariants were introduced in [1], [2]. A vertex $v \in V$ is *centroidal* if $\pi(v) = \pi(G)$, and the set of all centroidal vertices is the *centroid* of G.

Recently, these concepts and relations between them have been extensively studied (see [1], [2], [3], [4], [11], [12]). For example, in [3] the authors established the Nordhaus–Gaddum theorem for π and ρ. In [4] upper and lower bounds for π and ρ were obtained expressed in number n of vertices in G. Also, relations of both invariants with some other distance invariants (like diameter, radius, average eccentricity, average distance, etc.) were studied. The authors posed several conjectures (one of which was solved in [11]), among which the following.

Conjecture 1 *Among all connected graphs G on $n \geq 3$ vertices with average distance $\bar{l}$ and proximity π, the difference $\bar{l} - \pi$ is maximum for a graph G composed of three paths of almost equal lengths with a common end point.*

Conjecture 2 *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and average eccentricity ecc. Then*

$$ecc - \rho \leq \begin{cases} \frac{3n+1}{4}\frac{n-1}{n} - \frac{n}{2} & \text{if } n \text{ is odd,} \\ \frac{n-1}{4} - \frac{1}{4n-4} & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if G is a cycle C_n.

Conjecture 3 *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and radius r. Then*

$$\rho - r \geq \begin{cases} \frac{3-n}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4n-4} - \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The inequality is best possible as shown by the cycle C_n if n is even and by the graph composed by the cycle C_n together with two crossed edges on four successive vertices of the cycle.

In this paper we prove Conjecture 1, and find the extremal trees for $ecc - \rho$ and $\rho - r$ (maximal and minimal trees respectively) showing thus that Conjectures 2 and 3 hold for trees.

All these conjectures were obtained with the use of AutoGraphiX, a conjecture-making system in graph theory (see for example [6] and [7]). Also, some results on center and centroidal vertices will be used which are already known in literature since those concepts were also quite extensively studied (see for example [5], [8], [9], [10]).

2 Preliminaries

Let us introduce some additional notation for trees and state some auxiliary results known in literature. First, we will often use a notion of diametric path. So, if a tree G of diameter D has

diametric path P, we will suppose that vertices on P are denoted with v_i so that $P = v_0 v_1 \dots v_D$. When deleting edges of P from G, we obtain several connected components which are subtrees rooted in vertices of P. Now, G_i will denote a connected component of tree $G \backslash P$ rooted in vertex v_i of P and V_i will denote set of vertices of G_i.

Furthermore, for a tree G let $e \in E$ be an edge in G and $u \in V$ a vertex in G. With $G_u(e)$ we will denote the connected component of $G - e$ containing u. Also, we denote $V_u(e) = V(G_u(e))$ and $n_u(e) = |V_u(e)|$. Now the following lemma holds.

Lemma 4 *The following statements hold for a tree G:*

1. *a vertex $v \in V(G)$ is a centroidal vertex if and only if for any edge e incident with v holds $n_v(e) \geq \frac{n}{2}$,*
2. *G has at most two centroidal vertices,*
3. *if there are two centroidal vertices in G, then they are adjacent,*
4. *G has two centroidal vertices if and only if there is an edge e in G, such that the two components of $G - e$ have the same order. Furthermore, the end vertices of e are the two centroidal vertices of G.*

Proof. See [11]. ■

Also, we will often use transformation of tree G to G'. For the sake of notation simplicity, we will write D' for $D(G')$, ρ' for $\rho(G')$, $\pi'(v)$ for $\pi(v)$ in G', etc.

3 Average distance and proximity

To prove Conjecture 1 for trees, we will use graph transformations which transform tree to either:

1) path P_n,
2) a tree consisting of four paths of equal length with a common end point,
3) a tree consisting of three paths of almost equal length with a common end point.

So let us first prove that among those graphs the difference $\bar{l} - \pi$ is maximum for the last.

Lemma 5 *The difference $\bar{l} - \pi$ is greater for a tree G on n vertices consisting of three paths of almost equal length with a common end point than for path P_n.*

Proof. By direct calculation. ■

Lemma 6 *The difference $\bar{l} - \pi$ is greater for a tree G on n vertices consisting of three paths of almost equal length with a common end point than for a tree G' on n vertices consisting of four paths of equal length.*

Proof. First note that number of vertices n must be odd number, moreover $n = 4k + 1$. For a tree G' by a simple calculation we establish

$$\bar{l}(G') = \frac{3n^2 + 10n + 3}{16n}, \ \pi(G') = \frac{n + 3}{8}.$$

Since for a path P_n on odd number of vertices holds

$$\bar{l}(P_n) = \frac{n+1}{3},\ \pi(P_n) = \frac{n+1}{4},$$

it is easily verified that for $n \geq 9$ the difference $\bar{l} - \pi$ is greater or equal for P_n than for G', so the claim follows from Lemma 5. It only remains to prove the case $n = 5$, which is easily done by direct calculation. ■

Now, let us introduce a transformation of a general tree which decreases number of leafs in tree, but increases $\bar{l} - \pi$.

Lemma 7 *Let G be a tree on n vertices with at least four leafs. Then there is a tree G' on n vertices with three leafs for which the difference $\bar{l} - \pi$ is greater or equal than for G.*

Proof. Let u be centroidal vertex of G, let v be the branching vertex furthest from u. We distinguish two cases.

CASE I: $u \neq v$. Let G_v be a subtree of G rooted in v consisting of all vertices w such that path from u to w leads through v. Since v is branching vertex furthest from u, tree G_v consists of paths with a common end v. Let P_1 and P_2 be two such paths. For $i = 1, 2$ let x_i be a vertex in P_i adjacent to v and let y_i be a leaf in P_i. Let G' be a tree obtained from G by deleting edge vx_2 and adding edge x_2y_1. This transformation is illustrated in Figure 1. Note that G' has one leaf less than G. We want to prove that the difference $\bar{l} - \pi$ is greater for G' then for G. For that purpose let us denote $d_1 = d(v, y_1)$ and $d_2 = d(v, y_2)$. Note that

$$\pi(G') \leq \pi'(u) = \pi(u) + \frac{d_1 d_2}{n-1} = \pi(G) + \frac{d_1 d_2}{n-1}.$$

Also,

$$\bar{l}(G') = \bar{l}(G) + \frac{2}{n(n-1)} \cdot d_2(n - d_1 - d_2 - 1)d_1.$$

From here we obtain

$$\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G) + \frac{d_1 d_2}{n-1}\left(\frac{2(n - d_1 - d_2 - 1)}{n} - 1\right).$$

By Lemma 4 we have $n - d_1 - d_2 - 1 \geq \frac{n}{2}$, therefore $\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G)$.

CASE II: $u = v$. Obviously, v is the only branching vertex in G. Therefore G consists of paths with common end point v. Let P_1 and P_2 be two shortest such path. If $V \backslash (P_1 \cup P_2 \cup \{v\})$ contains at least $\frac{n}{2}$ vertices, then we make the same argument as in case I. Otherwise G is a tree consisting of four paths of equal length with a common end point and the claim follows by Lemma 6.

Applying the transformations from cases I and II repeatedly, one obtains the claimed. ■

Lemma 8 *Among trees with three leafs, the difference $\bar{l} - \pi$ is maximum for a tree G on n vertices consisting of three paths of almost equal length with a common end point.*

Proof. Let G be a tree with three leafs. That implies G consists of three paths with a common end vertex. Let u be centroidal vertex of G, let v be the branching vertex furthest from u. If $u \neq v$, then by the same argument as in case I of the proof of Lemma 7 we obtain that the difference $\bar{l} - \pi$ is greater or equal for path P_n than for G. Now the claimed follows from Lemma 5. Else if $u = v$,

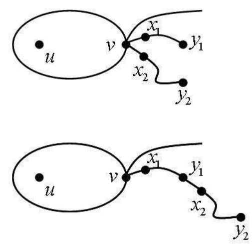

Figure 1: Tree transformation in the proof of Lemma 7.

then all three paths graph G consists of have less than $\frac{n}{2}$ vertices. Let v_1 be the leaf furthest from u and v_2 leaf closest to u. Let G' be a tree obtained from G by deleting edge incident to v_1 and adding edge v_1v_2. We want to prove that the difference $\bar{l} - \pi$ is greater or equal for G' than for G. Let $d_1 = d(u, v_1)$ and $d_2 = d(u, v_2)$. We have

$$\pi'(u) = \pi(u) - \frac{d_1 - d_2 - 1}{n - 1}.$$

Also

$$\bar{l}(G') = \bar{l}(G') - \frac{2}{n(n-1)} (n - d_1 - d_2 - 1)(d_1 - d_2 - 1).$$

From here we obtain

$$\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G) + \frac{d_1 - d_2 - 1}{n - 1} \left(1 - \frac{2}{n}(n - d_1 - d_2 - 1)\right).$$

Since all three paths G consists of have less then $\frac{n}{2}$ vertices, we can conclude that $n - d_1 - d_2 - 1 \leq \frac{n}{2}$ from which follows $\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G)$. By repeating this tree transformation we obtain the claim. ■

We can summarize the results of Lemmas 5, 6, 7 and 8 into following theorem.

Theorem 9 *Among all trees on $n \geq 3$ vertices with average distance $\bar{l}$ and proximity π, the difference $\bar{l} - \pi$ is maximum for a tree G composed of three paths of almost equal lengths with a common end point.*

Therefore, we have proved Conjecture 1 for trees. If for every graph we find a tree for which difference $\bar{l} - \pi$ is greater or equal, the Conjecture 1 for general graphs will follow from Theorem 9.

Theorem 10 *Among all connected graphs G on $n \geq 3$ vertices with average distance $\bar{l}$ and proximity π, the difference $\bar{l} - \pi$ is maximum for a graph G composed of three paths of almost equal lengths with a common end point.*

Proof. Let G be a connected graph on $n \geq 3$ vertices and let $u \in V(G)$ be a vertex in G such that $\pi(u) = \pi(G)$. Let G' be a breadth-first search tree of G rooted at u. Obviously, $\pi(G) = \pi(u) = \pi'(u) \geq \pi(G')$. As for $\bar{l}$, by deleting edges from G distances between vertices can only increase, therefore $\bar{l}(G) \leq \bar{l}(G')$. Now we have $\bar{l}(G) - \pi(G) \leq \bar{l}(G') - \pi(G')$ and the claim follows from Theorem 9. ■

4 Average eccentricity and remoteness

Now, let us find maximal trees for $ecc - \rho$, proving thus that Conjecture 2 holds for trees.

Lemma 11 *Let G be a tree on n vertices with diameter D and let $P = v_0 v_1 \ldots v_D$ be a diametric path in G. If there is $j \leq D/2$ such that the degree of v_k is at most 2 for $k \geq j+1$, then the difference $ecc - \rho$ is greater or equal for path P_n than for G.*

Proof. Let w be a leaf in G distinct from v_0 and v_D. Let G' be a tree obtained from G by deleting edge incident to w and adding edge $v_D w$. Note that G' has diameter $D+1$. We want to prove that difference $ecc - \rho$ did not decrease by this transformation. First note that eccentricity increased by 1 for at least $n - \frac{D+1}{2}$ vertices. Therefore, $ecc' \geq ecc + \frac{2n-D-1}{2n}$. As for remoteness, first note that $\rho(G) = \pi(v_D)$ and $\rho(G') = \pi'(w)$. Now, let d_w be the distance between vertices w and v_D in G, i.e. $d_w = d(w, v_D)$. Obviously $d_w \geq \frac{D+2}{2}$. Now, we have

$$\pi'(w) = \pi(v_D) + \frac{n - d_w - 1}{n-1} \leq \pi(v_D) + \frac{2n - D - 3}{2(n-1)}.$$

Therefore,

$$ecc' - \rho' \geq ecc - \rho + \frac{2n - D - 1}{2n} - \frac{2n - D - 3}{2(n-1)} \geq ecc - \rho.$$

Repeating this transformation, we obtain the claim. ■

Theorem 12 *Among trees on $n \geq 3$ vertices, the difference $ecc - \rho$ is maximum for path P_n.*

Proof. Let G be a tree on n vertices and diameter D. Let $P = v_0 v_1 \ldots v_D$ be a diametric path in G. Let G_i be a tree that is connected component of $G \backslash P$ rooted in v_i and let V_i be the vertex set of G_i. If there is $j \leq D/2$ such that the degree of v_k is at most 2 for $k \geq j+1$, then the claim follows from Lemma 11. Else, let v_j and v_k be vertices on P of degree at least 3 such that $j \leq \frac{D}{2} < k$ and $k - j$ is minimum possible. Let w_j be a vertex outside of P adjacent to v_j and let w_k be a vertex outside of P adjacent to v_k. Let G' be a tree obtained from G so that:

1) for every vertex w adjacent to v_j, except $w = w_j$ and $w = v_{j+1}$, edge wv_j is deleted and edge ww_j aded,

2) for every vertex w adjacent to v_k, except $w = w_k$ and $w = v_{k-1}$, edge wv_k is deleted and edge ww_k aded.

This transformation is illustrated with Figure 2. Note that diameter of G' equals $D+2$. We want to prove that $ecc' - \rho' \geq ecc - \rho$. For that purpose, let us denote

$$V_j' = \{v \in V_j : d(v, w_j) < d(v, v_j)\},$$
$$V_k' = \{v \in V_k : d(v, w_k) < d(v, v_k)\}.$$

Now, let us introduce following partition of set of vertices V

$$X_1 = V_0 \cup \ldots \cup V_{j-1} \cup (V_j \backslash (V_j' \cup \{v_j\})),$$
$$X_2 = V_j',$$
$$X_3 = \{v_j\} \cup V_{j+1} \cup \ldots \cup V_{k-1} \cup \{v_k\},$$
$$X_4 = V_k',$$
$$X_5 = (V_k \backslash (V_k' \cup \{v_k\})) \cup V_{k+1} \cup \ldots \cup V_D.$$

Let $x_i = |X_i|$. Now, let us compare $e'(v)$ and $e(v)$ for every vertex $v \in V$. Note that for $v \in X_2 \cup X_3 \cup X_4$ holds $e'(v) = e(v) + 1$, while for $v \in X_1 \cup X_5$ holds $e'(v) = e(v) + 2$. Therefore,

$$ecc' = ecc + \frac{2x_1 + x_2 + x_3 + x_4 + 2x_5}{n} = ecc + \Delta_1.$$

Now, we want to compare $\pi'(v)$ and $\pi(v)$ for every $v \in V$. We distinguish several cases depending whether $v \in X_1$, $v \in X_2$, $v \in X_3$, $v \in X_4$ or $v \in X_5$. It is sufficient to consider cases $v \in X_1$, $v \in X_2$ and $v \in X_3$, since $v \in X_4$ is analogous to $v \in X_2$ and $v \in X_5$ is analogous to $v \in X_1$.

If $v \in X_1$, then the difference $d'(v,u) - d(v,u)$ equals 0 for $u \in X_1$, equals -1 for $u \in X_2$, equals 1 for $u \in X_3 \cup X_4$ and equals 2 for $u \in X_5$. Therefore,

$$\pi'(v) = \pi(v) + \frac{-x_2 + x_3 + x_4 + 2x_5}{n-1} = \pi(v) + \Delta_2.$$

If $v \in X_2$, then the difference $d'(v,u) - d(v,u)$ equals -1 for $u \in X_1$, equals 0 for $u \in X_2 \cup X_3 \cup X_4$ and equals 1 for $u \in X_5$. Therefore,

$$\pi'(v) = \pi(v) + \frac{-x_1 + x_5}{n-1} = \pi(v) + \Delta_3.$$

If $v \in X_3$, then the difference $d'(v,u) - d(v,u)$ equals 1 for $u \in X_1 \cup X_5$ and equals 0 for $u \in X_2 \cup X_3 \cup X_4$. Therefore,

$$\pi'(v) = \pi(v) + \frac{x_1 + x_5}{n-1} = \pi(v) + \Delta_4.$$

It is easily verified that $\Delta_1 - \Delta_2 \geq 0$, $\Delta_1 - \Delta_3 \geq 0$ and $\Delta_1 - \Delta_4 \geq 0$, so for every $v \in V$ we obtain $ecc' - \pi'(v) \geq ecc - \pi(v)$.

Now, let $u \in V$ be a vertex for which $\pi'(u) = \rho(G')$. We have

$$\begin{aligned} ecc(G') - \rho(G') &= ecc(G') - \pi'(u) \geq ecc(G) - \pi(u) \geq \\ &\geq ecc(G) - \max\{\pi(v) : v \in V\} = ecc(G) - \rho(G). \end{aligned}$$

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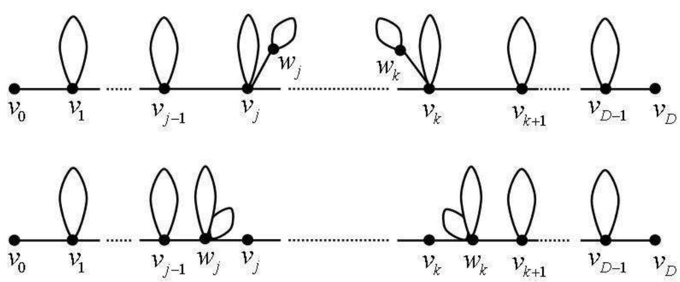

Figure 2: Tree transformation in the proof of Theorem 12.

Therefore, we have proved that P_n is the tree which maximizes the difference $ecc - \rho$. Now, from

$$ecc(P_n) - \rho(P_n) = \begin{cases} \frac{n-2}{4} & \text{for even } n, \\ \frac{n}{4} - \frac{2n+1}{4n} & \text{for odd } n. \end{cases}$$

easily follows that Conjecture 2 holds for trees.

5 Remoteness and radius

First, we want to find minimal trees for $\rho - r$. For that purpose, first step is to reduce the problem to caterpillar trees.

Lemma 13 *Let G be a tree on n vertices. There is a caterpillar tree G' on n vertices for which the difference $\rho - r$ is less or equal than for G.*

Proof. Let $P = v_0 v_1 \dots v_D$ be a diametric path in G. Let G_i be a tree that is connected component of $G \backslash P$ rooted in v_i and let V_i be the vertex set of G_i. Let G' be a caterpillar tree obtained from G in a following manner. In a tree G_i let v be the non-leaf vertex furthest from v_i, let $w_1, \dots, w_k$ be all leafs adjacent to v, and let u be the only remaining vertex adjacent to v. Now, for every $j = 1, \dots, k$ edge $w_j v$ is deleted and edge $w_j u$ is added. This transformation is illustrated with Figure 3.The procedure is done repeatedly in every G_i $(2 \leq i \leq D-2)$ until the caterpillar tree G' is obtained. Note that G' has the same diameter (and therefore radius) as G. What remains to be proved is that remoteness in G' is less or equal than in G. It is sufficient to prove that the described transformation does not increase remoteness. Obviously, $\pi'(u) \leq \pi(u)$ for every $u \in V \backslash \{v\}$. Number $\pi'(v)$ can be greater than $\pi(v)$, but note that $\pi'(v) = \pi'(w_i) \leq \pi(w_i) \leq \rho$. Therefore $\rho' \leq \rho$. ■

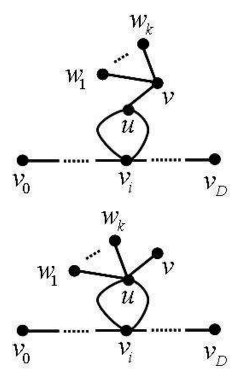

Figure 3: Tree transformation in the proof of Lemma 13.

Now that we reduced the problem to caterpillar trees, let us prove some auxiliary results for such trees. First note that because of Lemma 4, a leaf in a tree can not be centroidal vertex. Therefore, in a caterpillar tree a centroidal vertex must be on diametric path P.

Lemma 14 *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and only one centroidal vertex. Let $P = v_0 v_1 \dots v_D$ be a diametric path in G such that $v_j \in P$ is the only centroidal vertex in G and every of the vertices $v_{j+1}, \dots, v_D$ is of degree at most 2. Then there is a caterpillar tree G' on n vertices of diameter $D+1$ and remoteness at most $\rho + \frac{1}{2}$.*

Proof. If v_j is of degree 2, then by Lemma 4 follows that $j \leq \frac{D}{2}$, so the $\rho = \pi(v_D)$. Let w be any leaf in G distinct from v_0 and v_D. Let G' be a graph obtained from G by first deleting edge incident to w, then deleting edge $v_{j-1}v_j$ and adding path $v_{j-1}wv_j$ instead. This transformation is illustrated in Figure 4. Note that diameter of G' is $D+1$, while remoteness is still obtained

for v_D. Note that distances from v_D have increased by 1 for at most $\frac{n}{2}-1$ vertices. Therefore, $\pi'(v_D) \leq \pi(v_D) + \frac{n-2}{2(n-1)}$ from which follows $\rho' \leq \rho + \frac{1}{2}$ and the claim is proved in this case.

If degree of v_j is greater than 2, let w be a leaf on v_j, let $V_L = V_1 \cup \ldots \cup V_{j-1}$ and $V_R = V_{j+1} \cup \ldots \cup V_D$. Since v_j is centroidal vertex, from Lemma 4 follows that V_L and V_R have at most $\frac{n}{2}$ vertices. If any of them had exactly $\frac{n}{2}$ vertices, then G would have two centroidal vertices by Lemma 4, which would be contradiction with v_j being only centroidal vertex. Therefore, we conclude $|V_L| \leq \frac{n-1}{2}$ and $|V_R| \leq \frac{n-1}{2}$. Now it is possible to divide set of vertices $V_j \backslash \{v_j\}$ into two subsets V_j' and V_j'' such that $\left|V_L \cup V_j'\right| \leq \frac{n-1}{2}$ and $\left|V_R \cup V_j''\right| \leq \frac{n-1}{2}$. Let G' be a graph obtained from G by first deleting edge incident to w, then deleting edge v_jv_{j+1} and adding path v_jwv_{j+1} instead, and finally for every vertex $v \in V_j''$ edge vv_j is deleted and edge vw added. This transformation is illustrated in Figure 4. Note that diameter of G' is $D+1$. Now, if $v \in V_L \cup V_j' \cup \{v_j, w\}$ the distance $d(v,u)$ is increased by 1 only if $u \in V_R \cup V_j''$, therefore $\pi'(v) \leq \pi(v) + \frac{1}{2}$. If $v \in V_R \cup V_j''$ the distance $d(v,u)$ is increased by 1 only if $u \in V_L \cup V_j'$, therefore $\pi'(v) \leq \pi(v) + \frac{1}{2}$. We conclude $\rho' \leq \rho + \frac{1}{2}$, and the claim is proved in this case too. ■

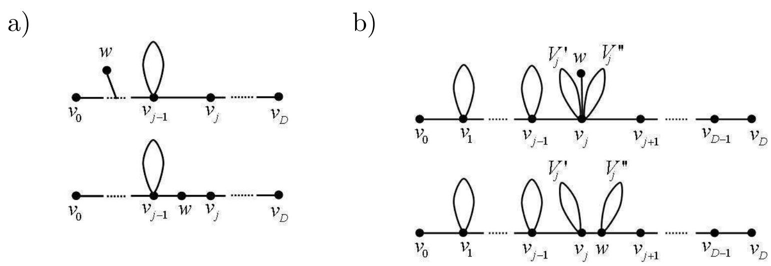

Figure 4: Tree transformations in the proof of Lemma 14: a) v_j is of degree 2, b) v_j is of degree at least 3.

Lemma 15 *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and exactly two centroidal vertices. Let $P = v_0v_1 \ldots v_D$ be a diametric path in G such that $v_j, v_{j+1} \in P$ are centroidal vertices and every of the vertices $v_{j+1}, \ldots, v_D$ is of degree at most 2. Then there is a caterpillar tree G' on n vertices of diameter $D+1$ and remoteness at most $\rho + \frac{1}{2}$.*

Proof. Since v_{j+1} is centroidal vertex, from Lemma 4 follows that $j \leq \frac{D}{2}$, so $\rho = \pi(v_D)$. Let w be any leaf in G distinct from v_0 and v_D. Let G' be a graph obtained from G by first deleting edge incident to w, then deleting edge v_jv_{j+1} and adding path v_jwv_{j+1} instead. The diameter of G' is $D+1$ and remoteness is still obtained for v_D. Note that distances from v_D increased by 1 for at most $\frac{n}{2}-1$ vertices, so $\pi'(v_D) \leq \pi(v_D) + \frac{n-2}{2(n-1)}$. Therefore, $\rho' \leq \rho + \frac{1}{2}$. ■

Lemma 16 *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and exactly two centroidal vertices of different degrees. Let $P = v_0v_1 \ldots v_D$ be a diametric path in G such that $v_j, v_{j+1} \in P$ are centroidal vertices and every of the vertices $v_0, \ldots, v_{j-1}, v_{j+2}, \ldots, v_D$ is of degree at most 2. Then there is a caterpillar tree G' on n vertices of diameter $D+1$ and remoteness at most $\rho + \frac{1}{2}$.*

Proof. Let $d_1 = d(v_0, v_j)$ and $d_2 = d(v_{j+1}, v_D)$. Without loss of generality we may assume that $d_1 \leq d_2$. Since the degrees of v_j and v_{j+1} differ, from Lemma 4 we conclude $d_1 \neq d_2$. Therefore,

$d_1 < d_2$. From this follows $j + 1 \leq \frac{D}{2}$, so $\rho = \pi(v_D)$. Let G' be a graph obtained from G so that for every leaf w incident to v_j (distinct from v_0) we delete edge wv_j and add edge wv_{j+1}. The diameter of G' is still D, while the remoteness ρ' is less or equal than ρ. Now the claim follows from Lemma 15 applied on G'. ■

Lemma 17 *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and exactly two centroidal vertices of equal degrees. Let $P = v_0 v_1 \dots v_D$ be a diametric path in G such that $v_j, v_{j+1} \in P$ are centroidal vertices and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2. Then the difference $\rho - r$ is less or equal for path P_n than for G.*

Proof. Let $d_1 = d(v_0, v_j)$ and $d_2 = d(v_{j+1}, v_D)$. Since v_j and v_{j+1} have equal degrees, and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2, we conclude that $d_1 = d_2$. Now, we will transform the tree twice which is illustrated with Figure 5. First, since G is not a path, both v_j and v_{j+1} must have a pendent leaf. Denote those leafs with w_1 and w_2 respectively. Let G' be a graph obtained from G by first deleting edges incident to w_1 and w_2, then deleting edge $v_j v_{j+1}$ and adding path $v_j w_1 w_2 v_{j+1}$ instead. Note that $D' = D + 2$. Therefore, $r' = r + 1$. Also, note that remoteness in both G and G' is obtained for v_0 and v_D. Since distances from v_0 have increased by 2 for at most $\frac{n}{2} - 1$ vertices, we conclude $\pi'(v_0) \leq \pi(v_0) + \frac{2(n-2)}{2(n-1)}$ from which follows $\rho' \leq \rho + 1$. Thus we obtain $\rho' - r' \leq \rho - r$. If G' is a path, then the claim is proved. Else, we transform G' so that for every leaf w in G' incident to v_j edge wv_j is deleted and edge ww_1 is added. Also, for every leaf w in G' incident to v_{j+1} edge wv_{j+1} is deleted and edge ww_2 is added. Note that this transformation changes neither radius neither remoteness. Thus we obtain the tree on which we can repeat the whole procedure. After repeating procedure finite number of times we obtain path P_n and the claim is proved. ■

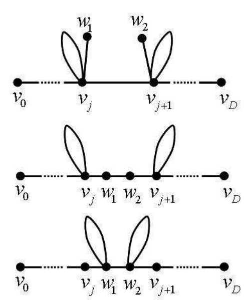

Figure 5: Tree transformations in the proof of Lemma 17.

Now that we have established auxiliary results for caterpillar trees, we can find minimal trees for $\rho - r$ among caterpillar trees.

Lemma 18 *Let G be a caterpillar tree on n vertices. If n is odd, then the difference $\rho - r$ is less or equal for path P_n then for G. If n is even, then the difference $\rho - r$ is less or equal for path P_{n-1} with a leaf appended to a central vertex than for G.*

Proof. Let D be the diameter in G and let $P = v_0 v_1 \dots v_D$ be the diametric path in G. Suppose $D \leq n - 3$. That means G has at least two leafs outside P. Let $v_j \in P$ be a centroidal vertex in

G. If there are two vertices v_k and v_l on P $(k < j < l)$ with a pendent leaf on them (distinct from v_0 and v_D), then the caterpillar tree G' obtained from G by deleting a leaf from v_j and v_k and adding a leaf on v_{j+1} and v_{k-1} has the same radius and the remoteness which is less or equal then in G. By repeating this procedure, we obtain a caterpillar tree G' of the same diameter as G with diametric path $P = v_0v_1 \dots v_D$ such that:

1. G' has exactly one centroidal vertex $v_j \in P$ and every of the vertices $v_{j+1}, \dots, v_D$ is of degree at most 2,
2. G' has two centroidal vertices $v_j, v_{j+1} \in P$ and every of the vertices $v_{j+1}, \dots, v_D$ is of degree at most 2,
3. G' has two centroidal vertices $v_j, v_{j+1} \in P$ and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2.

Therefore, on the obtained graph G' one of the Lemmas 14, 15, 16 or 17 can be applied. If Lemma 17 is applied, the claim is proved. Else if Lemma 14, 15 or 16 is applied, we obtain graph G' of diameter $D+1$ and remoteness $\rho + \frac{1}{2}$. Since for $D+1$ holds $D+1 \leq n-2$, we can apply the whole procedure with $G = G'$ (as the second step) and thus obtain a caterpillar tree G' of diameter $D+2$ and remoteness $\rho' \leq \rho + 1$. Since for thus obtained G' holds $D' = D + 2$, we conclude $r' = r + 1$. Therefore, $\rho' - r' \leq \rho - r$.

Repeating this double step, we obtain a caterpillar tree G' of diameter $D' = n-2$ or $D' = n-1$ for which the difference $\rho - r$ is less or equal than for G. Now we distinguish several cases with respect to D' and parity of n. Suppose first $D' = n-1$. Then $G' = P_n$. If n is odd then the claim is proved. If n is even it is easily verified that the difference $\rho - r$ is less for path P_{n-1} with a leaf appended to a central vertex than for $G' = P_n$ and the claim is proved in this case too. Suppose now that $D' = n-2$. That means G' is a path P_{n-1} with a leaf appended to one vertex of P_{n-1}. If n is odd, then deleting the only leaf in G' to extend it to P_n increases radius by 1 and remoteness by less than 1, so the claim holds. If n is even, then deleting the leaf in G' outside P_{n-1} and appending it to central vertex of P_{n-1} preserves the radius and decreases the remoteness. Therefore, the claim holds in this case too. ■

We can summarize results of these lemmas in the following theorem which gives minimal trees for $\rho - r$.

Theorem 19 *Let G be a tree on n vertices. If n is odd, then the difference $\rho - r$ is less or equal for path P_n then for G. If n is even, then the difference $\rho - r$ is less or equal for path P_{n-1} with a leaf appended to a central vertex than for G.*

Proof. Follows from Lemmas 13 and 18. ■

For a path P_n on odd number of vertices n holds $\rho - r = \frac{1}{2}$ which, together with Theorem 19, obviously implies that trees on odd number of vertices satisfy Conjecture 3. Now, let us consider graph G on even number of vertices n consisting of a path P_{n-1} with a leaf appended to a central vertex. For G holds $\rho - r = \frac{n}{2(n-1)}$ which implies that trees on even number of vertices satisfy Conjecture 3 too.

6 Conclusion

We have established that maximal tree for $\bar{l} - \pi$ is a tree composed of three paths of almost equal lengths with a common end point. Thus, we proved that Conjecture 1 posed in [4] for general graph

holds for trees. Using reduction of a graph to a corresponding subtree, this result enabled us to prove Conjecture 1 for general graphs too. Also, we established that maximal tree for $ecc - \rho$ is path P_n and that minimal tree for $\rho - r$ is path P_n in case of odd n and path P_{n-1} with a leaf appended to a central vertex in case of even n. Since for these extremal trees Conjectures 2 and 3 posed in [4] hold, it follows that those conjectures hold for trees.

7 Acknowledgements

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